



FHSST Authors

**The Free High School Science Texts:  
Textbooks for High School Students  
Studying the Sciences  
Mathematics  
Grades 10 - 12**

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# Contents

<b>I</b>	<b>Basics</b>	<b>1</b>
<b>1</b>	<b>Introduction to Book</b>	<b>3</b>
1.1	The Language of Mathematics . . . . .	3
<b>II</b>	<b>Grade 10</b>	<b>5</b>
<b>2</b>	<b>Review of Past Work</b>	<b>7</b>
2.1	Introduction . . . . .	7
2.2	What is a number? . . . . .	7
2.3	Sets . . . . .	7
2.4	Letters and Arithmetic . . . . .	8
2.5	Addition and Subtraction . . . . .	9
2.6	Multiplication and Division . . . . .	9
2.7	Brackets . . . . .	9
2.8	Negative Numbers . . . . .	10
2.8.1	What is a negative number? . . . . .	10
2.8.2	Working with Negative Numbers . . . . .	11
2.8.3	Living Without the Number Line . . . . .	12
2.9	Rearranging Equations . . . . .	13
2.10	Fractions and Decimal Numbers . . . . .	15
2.11	Scientific Notation . . . . .	16
2.12	Real Numbers . . . . .	16
2.12.1	Natural Numbers . . . . .	17
2.12.2	Integers . . . . .	17
2.12.3	Rational Numbers . . . . .	17
2.12.4	Irrational Numbers . . . . .	19
2.13	Mathematical Symbols . . . . .	20
2.14	Infinity . . . . .	20
2.15	End of Chapter Exercises . . . . .	21
<b>3</b>	<b>Rational Numbers - Grade 10</b>	<b>23</b>
3.1	Introduction . . . . .	23
3.2	The Big Picture of Numbers . . . . .	23
3.3	Definition . . . . .	23

3.4	Forms of Rational Numbers . . . . .	24
3.5	Converting Terminating Decimals into Rational Numbers . . . . .	25
3.6	Converting Repeating Decimals into Rational Numbers . . . . .	25
3.7	Summary . . . . .	26
3.8	End of Chapter Exercises . . . . .	27
<b>4</b>	<b>Exponentials - Grade 10</b>	<b>29</b>
4.1	Introduction . . . . .	29
4.2	Definition . . . . .	29
4.3	Laws of Exponents . . . . .	30
4.3.1	Exponential Law 1: $a^0 = 1$ . . . . .	30
4.3.2	Exponential Law 2: $a^m \times a^n = a^{m+n}$ . . . . .	30
4.3.3	Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$ . . . . .	31
4.3.4	Exponential Law 4: $a^m \div a^n = a^{m-n}$ . . . . .	32
4.3.5	Exponential Law 5: $(ab)^n = a^n b^n$ . . . . .	32
4.3.6	Exponential Law 6: $(a^m)^n = a^{mn}$ . . . . .	33
4.4	End of Chapter Exercises . . . . .	34
<b>5</b>	<b>Estimating Surds - Grade 10</b>	<b>37</b>
5.1	Introduction . . . . .	37
5.2	Drawing Surds on the Number Line (Optional) . . . . .	38
5.3	End of Chapter Exercises . . . . .	39
<b>6</b>	<b>Irrational Numbers and Rounding Off - Grade 10</b>	<b>41</b>
6.1	Introduction . . . . .	41
6.2	Irrational Numbers . . . . .	41
6.3	Rounding Off . . . . .	42
6.4	End of Chapter Exercises . . . . .	43
<b>7</b>	<b>Number Patterns - Grade 10</b>	<b>45</b>
7.1	Common Number Patterns . . . . .	45
7.1.1	Special Sequences . . . . .	46
7.2	Make your own Number Patterns . . . . .	46
7.3	Notation . . . . .	47
7.3.1	Patterns and Conjecture . . . . .	49
7.4	Exercises . . . . .	50
<b>8</b>	<b>Finance - Grade 10</b>	<b>53</b>
8.1	Introduction . . . . .	53
8.2	Foreign Exchange Rates . . . . .	53
8.2.1	How much is R1 really worth? . . . . .	53
8.2.2	Cross Currency Exchange Rates . . . . .	56
8.2.3	Enrichment: Fluctuating exchange rates . . . . .	57
8.3	Being Interested in Interest . . . . .	58



8.4	Simple Interest . . . . .	59
8.4.1	Other Applications of the Simple Interest Formula . . . . .	61
8.5	Compound Interest . . . . .	63
8.5.1	Fractions add up to the Whole . . . . .	65
8.5.2	The Power of Compound Interest . . . . .	65
8.5.3	Other Applications of Compound Growth . . . . .	67
8.6	Summary . . . . .	68
8.6.1	Definitions . . . . .	68
8.6.2	Equations . . . . .	68
8.7	End of Chapter Exercises . . . . .	69
<b>9</b>	<b>Products and Factors - Grade 10</b>	<b>71</b>
9.1	Introduction . . . . .	71
9.2	Recap of Earlier Work . . . . .	71
9.2.1	Parts of an Expression . . . . .	71
9.2.2	Product of Two Binomials . . . . .	71
9.2.3	Factorisation . . . . .	72
9.3	More Products . . . . .	74
9.4	Factorising a Quadratic . . . . .	76
9.5	Factorisation by Grouping . . . . .	79
9.6	Simplification of Fractions . . . . .	80
9.7	End of Chapter Exercises . . . . .	82
<b>10</b>	<b>Equations and Inequalities - Grade 10</b>	<b>83</b>
10.1	Strategy for Solving Equations . . . . .	83
10.2	Solving Linear Equations . . . . .	84
10.3	Solving Quadratic Equations . . . . .	89
10.4	Exponential Equations of the form $ka^{(x+p)} = m$ . . . . .	93
10.4.1	Algebraic Solution . . . . .	93
10.5	Linear Inequalities . . . . .	96
10.6	Linear Simultaneous Equations . . . . .	99
10.6.1	Finding solutions . . . . .	99
10.6.2	Graphical Solution . . . . .	99
10.6.3	Solution by Substitution . . . . .	101
10.7	Mathematical Models . . . . .	103
10.7.1	Introduction . . . . .	103
10.7.2	Problem Solving Strategy . . . . .	104
10.7.3	Application of Mathematical Modelling . . . . .	104
10.7.4	End of Chapter Exercises . . . . .	106
10.8	Introduction to Functions and Graphs . . . . .	107
10.9	Functions and Graphs in the Real-World . . . . .	107
10.10	Recap . . . . .	107

10.10.1 Variables and Constants . . . . .	107
10.10.2 Relations and Functions . . . . .	108
10.10.3 The Cartesian Plane . . . . .	108
10.10.4 Drawing Graphs . . . . .	109
10.10.5 Notation used for Functions . . . . .	110
10.11 Characteristics of Functions - All Grades . . . . .	112
10.11.1 Dependent and Independent Variables . . . . .	112
10.11.2 Domain and Range . . . . .	113
10.11.3 Intercepts with the Axes . . . . .	113
10.11.4 Turning Points . . . . .	114
10.11.5 Asymptotes . . . . .	114
10.11.6 Lines of Symmetry . . . . .	114
10.11.7 Intervals on which the Function Increases/Decreases . . . . .	114
10.11.8 Discrete or Continuous Nature of the Graph . . . . .	114
10.12 Graphs of Functions . . . . .	116
10.12.1 Functions of the form $y = ax + q$ . . . . .	116
10.12.2 Functions of the Form $y = ax^2 + q$ . . . . .	120
10.12.3 Functions of the Form $y = \frac{a}{x} + q$ . . . . .	125
10.12.4 Functions of the Form $y = ab^{(x)} + q$ . . . . .	129
10.13 End of Chapter Exercises . . . . .	133
<b>11 Average Gradient - Grade 10 Extension</b>	<b>135</b>
11.1 Introduction . . . . .	135
11.2 Straight-Line Functions . . . . .	135
11.3 Parabolic Functions . . . . .	136
11.4 End of Chapter Exercises . . . . .	138
<b>12 Geometry Basics</b>	<b>139</b>
12.1 Introduction . . . . .	139
12.2 Points and Lines . . . . .	139
12.3 Angles . . . . .	140
12.3.1 Measuring angles . . . . .	141
12.3.2 Special Angles . . . . .	141
12.3.3 Special Angle Pairs . . . . .	143
12.3.4 Parallel Lines intersected by Transversal Lines . . . . .	143
12.4 Polygons . . . . .	147
12.4.1 Triangles . . . . .	147
12.4.2 Quadrilaterals . . . . .	152
12.4.3 Other polygons . . . . .	155
12.4.4 Extra . . . . .	156
12.5 Exercises . . . . .	157
12.5.1 Challenge Problem . . . . .	159

<b>13 Geometry - Grade 10</b>	<b>161</b>
13.1 Introduction . . . . .	161
13.2 Right Prisms and Cylinders . . . . .	161
13.2.1 Surface Area . . . . .	162
13.2.2 Volume . . . . .	164
13.3 Polygons . . . . .	167
13.3.1 Similarity of Polygons . . . . .	167
13.4 Co-ordinate Geometry . . . . .	171
13.4.1 Introduction . . . . .	171
13.4.2 Distance between Two Points . . . . .	172
13.4.3 Calculation of the Gradient of a Line . . . . .	173
13.4.4 Midpoint of a Line . . . . .	174
13.5 Transformations . . . . .	177
13.5.1 Translation of a Point . . . . .	177
13.5.2 Reflection of a Point . . . . .	179
13.6 End of Chapter Exercises . . . . .	185
<b>14 Trigonometry - Grade 10</b>	<b>189</b>
14.1 Introduction . . . . .	189
14.2 Where Trigonometry is Used . . . . .	190
14.3 Similarity of Triangles . . . . .	190
14.4 Definition of the Trigonometric Functions . . . . .	191
14.5 Simple Applications of Trigonometric Functions . . . . .	195
14.5.1 Height and Depth . . . . .	195
14.5.2 Maps and Plans . . . . .	197
14.6 Graphs of Trigonometric Functions . . . . .	199
14.6.1 Graph of $\sin \theta$ . . . . .	199
14.6.2 Functions of the form $y = a \sin(x) + q$ . . . . .	200
14.6.3 Graph of $\cos \theta$ . . . . .	202
14.6.4 Functions of the form $y = a \cos(x) + q$ . . . . .	202
14.6.5 Comparison of Graphs of $\sin \theta$ and $\cos \theta$ . . . . .	204
14.6.6 Graph of $\tan \theta$ . . . . .	204
14.6.7 Functions of the form $y = a \tan(x) + q$ . . . . .	205
14.7 End of Chapter Exercises . . . . .	208
<b>15 Statistics - Grade 10</b>	<b>211</b>
15.1 Introduction . . . . .	211
15.2 Recap of Earlier Work . . . . .	211
15.2.1 Data and Data Collection . . . . .	211
15.2.2 Methods of Data Collection . . . . .	212
15.2.3 Samples and Populations . . . . .	213
15.3 Example Data Sets . . . . .	213

15.3.1 Data Set 1: Tossing a Coin . . . . .	213
15.3.2 Data Set 2: Casting a die . . . . .	213
15.3.3 Data Set 3: Mass of a Loaf of Bread . . . . .	214
15.3.4 Data Set 4: Global Temperature . . . . .	214
15.3.5 Data Set 5: Price of Petrol . . . . .	215
15.4 Grouping Data . . . . .	215
15.4.1 Exercises - Grouping Data . . . . .	216
15.5 Graphical Representation of Data . . . . .	217
15.5.1 Bar and Compound Bar Graphs . . . . .	217
15.5.2 Histograms and Frequency Polygons . . . . .	217
15.5.3 Pie Charts . . . . .	219
15.5.4 Line and Broken Line Graphs . . . . .	220
15.5.5 Exercises - Graphical Representation of Data . . . . .	221
15.6 Summarising Data . . . . .	222
15.6.1 Measures of Central Tendency . . . . .	222
15.6.2 Measures of Dispersion . . . . .	225
15.6.3 Exercises - Summarising Data . . . . .	228
15.7 Misuse of Statistics . . . . .	229
15.7.1 Exercises - Misuse of Statistics . . . . .	230
15.8 Summary of Definitions . . . . .	232
15.9 Exercises . . . . .	232
<b>16 Probability - Grade 10</b>	<b>235</b>
16.1 Introduction . . . . .	235
16.2 Random Experiments . . . . .	235
16.2.1 Sample Space of a Random Experiment . . . . .	235
16.3 Probability Models . . . . .	238
16.3.1 Classical Theory of Probability . . . . .	239
16.4 Relative Frequency vs. Probability . . . . .	240
16.5 Project Idea . . . . .	242
16.6 Probability Identities . . . . .	242
16.7 Mutually Exclusive Events . . . . .	243
16.8 Complementary Events . . . . .	244
16.9 End of Chapter Exercises . . . . .	246
<b>III Grade 11</b>	<b>249</b>
<b>17 Exponents - Grade 11</b>	<b>251</b>
17.1 Introduction . . . . .	251
17.2 Laws of Exponents . . . . .	251
17.2.1 Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ . . . . .	251
17.3 Exponentials in the Real-World . . . . .	253
17.4 End of chapter Exercises . . . . .	254

<b>18 Surds - Grade 11</b>	<b>255</b>
18.1 Surd Calculations . . . . .	255
18.1.1 Surd Law 1: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ . . . . .	255
18.1.2 Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ . . . . .	255
18.1.3 Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ . . . . .	256
18.1.4 Like and Unlike Surds . . . . .	256
18.1.5 Simplest Surd form . . . . .	257
18.1.6 Rationalising Denominators . . . . .	258
18.2 End of Chapter Exercises . . . . .	259
<b>19 Error Margins - Grade 11</b>	<b>261</b>
<b>20 Quadratic Sequences - Grade 11</b>	<b>265</b>
20.1 Introduction . . . . .	265
20.2 What is a <i>quadratic sequence</i> ? . . . . .	265
20.3 End of chapter Exercises . . . . .	269
<b>21 Finance - Grade 11</b>	<b>271</b>
21.1 Introduction . . . . .	271
21.2 Depreciation . . . . .	271
21.3 Simple Depreciation (it really is simple!) . . . . .	271
21.4 Compound Depreciation . . . . .	274
21.5 Present Values or Future Values of an Investment or Loan . . . . .	276
21.5.1 Now or Later . . . . .	276
21.6 Finding $i$ . . . . .	278
21.7 Finding $n$ - Trial and Error . . . . .	279
21.8 Nominal and Effective Interest Rates . . . . .	280
21.8.1 The General Formula . . . . .	281
21.8.2 De-coding the Terminology . . . . .	282
21.9 Formulae Sheet . . . . .	284
21.9.1 Definitions . . . . .	284
21.9.2 Equations . . . . .	285
21.10 End of Chapter Exercises . . . . .	285
<b>22 Solving Quadratic Equations - Grade 11</b>	<b>287</b>
22.1 Introduction . . . . .	287
22.2 Solution by Factorisation . . . . .	287
22.3 Solution by Completing the Square . . . . .	290
22.4 Solution by the Quadratic Formula . . . . .	293
22.5 Finding an equation when you know its roots . . . . .	296
22.6 End of Chapter Exercises . . . . .	299

<b>23 Solving Quadratic Inequalities - Grade 11</b>	<b>301</b>
23.1 Introduction . . . . .	301
23.2 Quadratic Inequalities . . . . .	301
23.3 End of Chapter Exercises . . . . .	304
<b>24 Solving Simultaneous Equations - Grade 11</b>	<b>307</b>
24.1 Graphical Solution . . . . .	307
24.2 Algebraic Solution . . . . .	309
<b>25 Mathematical Models - Grade 11</b>	<b>313</b>
25.1 Real-World Applications: Mathematical Models . . . . .	313
25.2 End of Chapter Exercises . . . . .	317
<b>26 Quadratic Functions and Graphs - Grade 11</b>	<b>321</b>
26.1 Introduction . . . . .	321
26.2 Functions of the Form $y = a(x + p)^2 + q$ . . . . .	321
26.2.1 Domain and Range . . . . .	322
26.2.2 Intercepts . . . . .	323
26.2.3 Turning Points . . . . .	324
26.2.4 Axes of Symmetry . . . . .	325
26.2.5 Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$ . . . . .	325
26.2.6 Writing an equation of a shifted parabola . . . . .	327
26.3 End of Chapter Exercises . . . . .	327
<b>27 Hyperbolic Functions and Graphs - Grade 11</b>	<b>329</b>
27.1 Introduction . . . . .	329
27.2 Functions of the Form $y = \frac{a}{x+p} + q$ . . . . .	329
27.2.1 Domain and Range . . . . .	330
27.2.2 Intercepts . . . . .	331
27.2.3 Asymptotes . . . . .	332
27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$ . . . . .	333
27.3 End of Chapter Exercises . . . . .	333
<b>28 Exponential Functions and Graphs - Grade 11</b>	<b>335</b>
28.1 Introduction . . . . .	335
28.2 Functions of the Form $y = ab^{(x+p)} + q$ . . . . .	335
28.2.1 Domain and Range . . . . .	336
28.2.2 Intercepts . . . . .	337
28.2.3 Asymptotes . . . . .	338
28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$ . . . . .	338
28.3 End of Chapter Exercises . . . . .	339
<b>29 Gradient at a Point - Grade 11</b>	<b>341</b>
29.1 Introduction . . . . .	341
29.2 Average Gradient . . . . .	341
29.3 End of Chapter Exercises . . . . .	344

<b>30 Linear Programming - Grade 11</b>	<b>345</b>
30.1 Introduction . . . . .	345
30.2 Terminology . . . . .	345
30.2.1 Decision Variables . . . . .	345
30.2.2 Objective Function . . . . .	345
30.2.3 Constraints . . . . .	346
30.2.4 Feasible Region and Points . . . . .	346
30.2.5 The Solution . . . . .	346
30.3 Example of a Problem . . . . .	347
30.4 Method of Linear Programming . . . . .	347
30.5 Skills you will need . . . . .	347
30.5.1 Writing Constraint Equations . . . . .	347
30.5.2 Writing the Objective Function . . . . .	348
30.5.3 Solving the Problem . . . . .	350
30.6 End of Chapter Exercises . . . . .	352
<b>31 Geometry - Grade 11</b>	<b>357</b>
31.1 Introduction . . . . .	357
31.2 Right Pyramids, Right Cones and Spheres . . . . .	357
31.3 Similarity of Polygons . . . . .	360
31.4 Triangle Geometry . . . . .	361
31.4.1 Proportion . . . . .	361
31.5 Co-ordinate Geometry . . . . .	368
31.5.1 Equation of a Line between Two Points . . . . .	368
31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line . . . . .	371
31.5.3 Inclination of a Line . . . . .	371
31.6 Transformations . . . . .	373
31.6.1 Rotation of a Point . . . . .	373
31.6.2 Enlargement of a Polygon 1 . . . . .	376
<b>32 Trigonometry - Grade 11</b>	<b>381</b>
32.1 History of Trigonometry . . . . .	381
32.2 Graphs of Trigonometric Functions . . . . .	381
32.2.1 Functions of the form $y = \sin(k\theta)$ . . . . .	381
32.2.2 Functions of the form $y = \cos(k\theta)$ . . . . .	383
32.2.3 Functions of the form $y = \tan(k\theta)$ . . . . .	384
32.2.4 Functions of the form $y = \sin(\theta + p)$ . . . . .	385
32.2.5 Functions of the form $y = \cos(\theta + p)$ . . . . .	386
32.2.6 Functions of the form $y = \tan(\theta + p)$ . . . . .	387
32.3 Trigonometric Identities . . . . .	389
32.3.1 Deriving Values of Trigonometric Functions for $30^\circ$ , $45^\circ$ and $60^\circ$ . . . . .	389
32.3.2 Alternate Definition for $\tan \theta$ . . . . .	391

32.3.3	A Trigonometric Identity . . . . .	392
32.3.4	Reduction Formula . . . . .	394
32.4	Solving Trigonometric Equations . . . . .	399
32.4.1	Graphical Solution . . . . .	399
32.4.2	Algebraic Solution . . . . .	401
32.4.3	Solution using CAST diagrams . . . . .	403
32.4.4	General Solution Using Periodicity . . . . .	405
32.4.5	Linear Trigonometric Equations . . . . .	406
32.4.6	Quadratic and Higher Order Trigonometric Equations . . . . .	406
32.4.7	More Complex Trigonometric Equations . . . . .	407
32.5	Sine and Cosine Identities . . . . .	409
32.5.1	The Sine Rule . . . . .	409
32.5.2	The Cosine Rule . . . . .	412
32.5.3	The Area Rule . . . . .	414
32.6	Exercises . . . . .	416
<b>33</b>	<b>Statistics - Grade 11</b>	<b>419</b>
33.1	Introduction . . . . .	419
33.2	Standard Deviation and Variance . . . . .	419
33.2.1	Variance . . . . .	419
33.2.2	Standard Deviation . . . . .	421
33.2.3	Interpretation and Application . . . . .	423
33.2.4	Relationship between Standard Deviation and the Mean . . . . .	424
33.3	Graphical Representation of Measures of Central Tendency and Dispersion . . . . .	424
33.3.1	Five Number Summary . . . . .	424
33.3.2	Box and Whisker Diagrams . . . . .	425
33.3.3	Cumulative Histograms . . . . .	426
33.4	Distribution of Data . . . . .	428
33.4.1	Symmetric and Skewed Data . . . . .	428
33.4.2	Relationship of the Mean, Median, and Mode . . . . .	428
33.5	Scatter Plots . . . . .	429
33.6	Misuse of Statistics . . . . .	432
33.7	End of Chapter Exercises . . . . .	435
<b>34</b>	<b>Independent and Dependent Events - Grade 11</b>	<b>437</b>
34.1	Introduction . . . . .	437
34.2	Definitions . . . . .	437
34.2.1	Identification of Independent and Dependent Events . . . . .	438
34.3	End of Chapter Exercises . . . . .	441
<b>IV</b>	<b>Grade 12</b>	<b>443</b>
<b>35</b>	<b>Logarithms - Grade 12</b>	<b>445</b>
35.1	Definition of Logarithms . . . . .	445



35.2	Logarithm Bases . . . . .	446
35.3	Laws of Logarithms . . . . .	447
35.4	Logarithm Law 1: $\log_a 1 = 0$ . . . . .	447
35.5	Logarithm Law 2: $\log_a(a) = 1$ . . . . .	448
35.6	Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$ . . . . .	448
35.7	Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ . . . . .	449
35.8	Logarithm Law 5: $\log_a(x^b) = b \log_a(x)$ . . . . .	450
35.9	Logarithm Law 6: $\log_a(\sqrt[b]{x}) = \frac{\log_a(x)}{b}$ . . . . .	450
35.10	Solving simple log equations . . . . .	452
35.10.1	Exercises . . . . .	454
35.11	Logarithmic applications in the Real World . . . . .	454
35.11.1	Exercises . . . . .	455
35.12	End of Chapter Exercises . . . . .	455
<b>36</b>	<b>Sequences and Series - Grade 12</b>	<b>457</b>
36.1	Introduction . . . . .	457
36.2	Arithmetic Sequences . . . . .	457
36.2.1	General Equation for the $n^{\text{th}}$ -term of an Arithmetic Sequence . . . . .	458
36.3	Geometric Sequences . . . . .	459
36.3.1	Example - A Flu Epidemic . . . . .	459
36.3.2	General Equation for the $n^{\text{th}}$ -term of a Geometric Sequence . . . . .	461
36.3.3	Exercises . . . . .	461
36.4	Recursive Formulae for Sequences . . . . .	462
36.5	Series . . . . .	463
36.5.1	Some Basics . . . . .	463
36.5.2	Sigma Notation . . . . .	463
36.6	Finite Arithmetic Series . . . . .	465
36.6.1	General Formula for a Finite Arithmetic Series . . . . .	466
36.6.2	Exercises . . . . .	467
36.7	Finite Squared Series . . . . .	468
36.8	Finite Geometric Series . . . . .	469
36.8.1	Exercises . . . . .	470
36.9	Infinite Series . . . . .	471
36.9.1	Infinite Geometric Series . . . . .	471
36.9.2	Exercises . . . . .	472
36.10	End of Chapter Exercises . . . . .	472
<b>37</b>	<b>Finance - Grade 12</b>	<b>477</b>
37.1	Introduction . . . . .	477
37.2	Finding the Length of the Investment or Loan . . . . .	477
37.3	A Series of Payments . . . . .	478
37.3.1	Sequences and Series . . . . .	479

37.3.2 Present Values of a series of Payments . . . . .	479
37.3.3 Future Value of a series of Payments . . . . .	484
37.3.4 Exercises - Present and Future Values . . . . .	485
37.4 Investments and Loans . . . . .	485
37.4.1 Loan Schedules . . . . .	485
37.4.2 Exercises - Investments and Loans . . . . .	489
37.4.3 Calculating Capital Outstanding . . . . .	489
37.5 Formulae Sheet . . . . .	489
37.5.1 Definitions . . . . .	490
37.5.2 Equations . . . . .	490
37.6 End of Chapter Exercises . . . . .	490
<b>38 Factorising Cubic Polynomials - Grade 12</b>	<b>493</b>
38.1 Introduction . . . . .	493
38.2 The Factor Theorem . . . . .	493
38.3 Factorisation of Cubic Polynomials . . . . .	494
38.4 Exercises - Using Factor Theorem . . . . .	496
38.5 Solving Cubic Equations . . . . .	496
38.5.1 Exercises - Solving of Cubic Equations . . . . .	498
38.6 End of Chapter Exercises . . . . .	498
<b>39 Functions and Graphs - Grade 12</b>	<b>501</b>
39.1 Introduction . . . . .	501
39.2 Definition of a Function . . . . .	501
39.2.1 Exercises . . . . .	501
39.3 Notation used for Functions . . . . .	502
39.4 Graphs of Inverse Functions . . . . .	502
39.4.1 Inverse Function of $y = ax + q$ . . . . .	503
39.4.2 Exercises . . . . .	504
39.4.3 Inverse Function of $y = ax^2$ . . . . .	504
39.4.4 Exercises . . . . .	504
39.4.5 Inverse Function of $y = a^x$ . . . . .	506
39.4.6 Exercises . . . . .	506
39.5 End of Chapter Exercises . . . . .	507
<b>40 Differential Calculus - Grade 12</b>	<b>509</b>
40.1 Why do I have to learn this stuff? . . . . .	509
40.2 Limits . . . . .	510
40.2.1 A Tale of Achilles and the Tortoise . . . . .	510
40.2.2 Sequences, Series and Functions . . . . .	511
40.2.3 Limits . . . . .	512
40.2.4 Average Gradient and Gradient at a Point . . . . .	516
40.3 Differentiation from First Principles . . . . .	519

40.4	Rules of Differentiation . . . . .	521
40.4.1	Summary of Differentiation Rules . . . . .	522
40.5	Applying Differentiation to Draw Graphs . . . . .	523
40.5.1	Finding Equations of Tangents to Curves . . . . .	523
40.5.2	Curve Sketching . . . . .	524
40.5.3	Local minimum, Local maximum and Point of Inflexion . . . . .	529
40.6	Using Differential Calculus to Solve Problems . . . . .	530
40.6.1	Rate of Change problems . . . . .	534
40.7	End of Chapter Exercises . . . . .	535
<b>41</b>	<b>Linear Programming - Grade 12</b>	<b>539</b>
41.1	Introduction . . . . .	539
41.2	Terminology . . . . .	539
41.2.1	Feasible Region and Points . . . . .	539
41.3	Linear Programming and the Feasible Region . . . . .	540
41.4	End of Chapter Exercises . . . . .	546
<b>42</b>	<b>Geometry - Grade 12</b>	<b>549</b>
42.1	Introduction . . . . .	549
42.2	Circle Geometry . . . . .	549
42.2.1	Terminology . . . . .	549
42.2.2	Axioms . . . . .	550
42.2.3	Theorems of the Geometry of Circles . . . . .	550
42.3	Co-ordinate Geometry . . . . .	566
42.3.1	Equation of a Circle . . . . .	566
42.3.2	Equation of a Tangent to a Circle at a Point on the Circle . . . . .	569
42.4	Transformations . . . . .	571
42.4.1	Rotation of a Point about an angle $\theta$ . . . . .	571
42.4.2	Characteristics of Transformations . . . . .	573
42.4.3	Characteristics of Transformations . . . . .	573
42.5	Exercises . . . . .	574
<b>43</b>	<b>Trigonometry - Grade 12</b>	<b>577</b>
43.1	Compound Angle Identities . . . . .	577
43.1.1	Derivation of $\sin(\alpha + \beta)$ . . . . .	577
43.1.2	Derivation of $\sin(\alpha - \beta)$ . . . . .	578
43.1.3	Derivation of $\cos(\alpha + \beta)$ . . . . .	578
43.1.4	Derivation of $\cos(\alpha - \beta)$ . . . . .	579
43.1.5	Derivation of $\sin 2\alpha$ . . . . .	579
43.1.6	Derivation of $\cos 2\alpha$ . . . . .	579
43.1.7	Problem-solving Strategy for Identities . . . . .	580
43.2	Applications of Trigonometric Functions . . . . .	582
43.2.1	Problems in Two Dimensions . . . . .	582

43.2.2 Problems in 3 dimensions . . . . .	584
43.3 Other Geometries . . . . .	586
43.3.1 Taxicab Geometry . . . . .	586
43.3.2 Manhattan distance . . . . .	586
43.3.3 Spherical Geometry . . . . .	587
43.3.4 Fractal Geometry . . . . .	588
43.4 End of Chapter Exercises . . . . .	589
<b>44 Statistics - Grade 12</b>	<b>591</b>
44.1 Introduction . . . . .	591
44.2 A Normal Distribution . . . . .	591
44.3 Extracting a Sample Population . . . . .	593
44.4 Function Fitting and Regression Analysis . . . . .	594
44.4.1 The Method of Least Squares . . . . .	596
44.4.2 Using a calculator . . . . .	597
44.4.3 Correlation coefficients . . . . .	599
44.5 Exercises . . . . .	600
<b>45 Combinations and Permutations - Grade 12</b>	<b>603</b>
45.1 Introduction . . . . .	603
45.2 Counting . . . . .	603
45.2.1 Making a List . . . . .	603
45.2.2 Tree Diagrams . . . . .	604
45.3 Notation . . . . .	604
45.3.1 The Factorial Notation . . . . .	604
45.4 The Fundamental Counting Principle . . . . .	604
45.5 Combinations . . . . .	605
45.5.1 Counting Combinations . . . . .	605
45.5.2 Combinatorics and Probability . . . . .	606
45.6 Permutations . . . . .	606
45.6.1 Counting Permutations . . . . .	607
45.7 Applications . . . . .	608
45.8 Exercises . . . . .	610
<b>V Exercises</b>	<b>613</b>
<b>46 General Exercises</b>	<b>615</b>
<b>47 Exercises - Not covered in Syllabus</b>	<b>617</b>
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# Chapter 17

## Exponents - Grade 11

### 17.1 Introduction

In Grade 10 we studied exponential numbers and learnt that there were six laws that made working with exponential numbers easier. There is one law that we did not study in Grade 10. This will be described here.

### 17.2 Laws of Exponents

In Grade 10, we worked only with indices that were integers. What happens when the index is not an integer, but is a rational number? This leads us to the final law of exponents,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad (17.1)$$

#### 17.2.1 Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

We say that  $x$  is an  $n$ th root of  $b$  if  $x^n = b$ . For example,  $(-1)^4 = 1$ , so  $-1$  is a 4th root of 1. Using law 6, we notice that

$$\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n} \times n} = a^m \quad (17.2)$$

therefore  $a^{\frac{m}{n}}$  must be an  $n$ th root of  $a^m$ . We can therefore say

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad (17.3)$$

where  $\sqrt[n]{a^m}$  is the  $n$ th root of  $a^m$  (if it exists).

For example,

$$2^{\frac{2}{3}} = \sqrt[3]{2^2}$$

A number may not always have a real  $n$ th root. For example, if  $n = 2$  and  $a = -1$ , then there is no real number such that  $x^2 = -1$  because  $x^2$  can never be a negative number.



#### *Extension: Complex Numbers*

There are numbers which can solve problems like  $x^2 = -1$ , but they are beyond the scope of this book. They are called *complex numbers*.

It is also possible for more than one  $n$ th root of a number to exist. For example,  $(-2)^2 = 4$  and  $2^2 = 4$ , so both  $-2$  and  $2$  are 2nd (square) roots of 4. Usually if there is more than one root, we choose the positive real solution and move on.



### Worked Example 81: Rational Exponents

**Question:** Simplify without using a calculator:

$$\left(\frac{5}{4^{-1} - 9^{-1}}\right)^{\frac{1}{2}}$$

**Answer**

**Step 1 :** Rewrite negative exponents as numbers with positive indices

$$= \left(\frac{5}{\frac{1}{4} - \frac{1}{9}}\right)^{\frac{1}{2}}$$

**Step 2 :** Simplify inside brackets

$$\begin{aligned} &= \left(\frac{5}{1} \div \frac{9-4}{36}\right)^{\frac{1}{2}} \\ &= \left(\frac{5}{1} \times \frac{36}{5}\right)^{\frac{1}{2}} \\ &= (6^2)^{\frac{1}{2}} \end{aligned}$$

**Step 3 :** Apply exponential law 6

$$= 6$$



### Worked Example 82: More rational Exponents

**Question:** Simplify:

$$(16x^4)^{\frac{3}{4}}$$

**Answer**

**Step 1 :** Covert the number co-efficient to index-form with a prime base

$$= (2^4 x^4)^{\frac{3}{4}}$$

**Step 2 :** Apply exponential laws

$$\begin{aligned} &= 2^{4 \times \frac{3}{4}} \cdot x^{4 \times \frac{3}{4}} \\ &= 2^3 \cdot x^3 \\ &= 8x^3 \end{aligned}$$

**Exercise: Applying laws**

Use all the laws to:

1. Simplify:

(a)  $(x^0) + 5x^0 - (0,25)^{-0,5} + 8^{\frac{2}{3}}$

(b)  $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$

(c)  $\frac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$

(d)  $(64m^6)^{\frac{2}{3}}$

2. Re-write the expression as a power of  $x$ :

$$x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$$

## 17.3 Exponentials in the Real-World

In Chapter 8, you used exponentials to calculate different types of interest, for example on a savings account or on a loan and compound growth.



### Worked Example 83: Exponentials in the Real world

**Question:** A type of bacteria has a very high exponential growth rate at 80% every hour. If there are 10 bacteria, determine how many there will be in 5 hours, in 1 day and in 1 week?

**Answer**

**Step 1:**  $Population = Initial\ population \times (1 + growth\ percentage)^{time\ period\ in\ hours}$

Therefore, in this case:

$Population = 10(1,8)^n$ , where  $n =$  number of hours

**Step 2 : In 5 hours**

$Population = 10(1,8)^5 = 188$

**Step 3 : In 1 day = 24 hours**

$Population = 10(1,8)^{24} = 13\ 382\ 588$

**Step 4 : in 1 week = 168 hours**

$Population = 10(1,8)^{168} = 7,687 \times 10^{43}$

Note this answer is given in scientific notation as it is a very big number.



### Worked Example 84: More Exponentials in the Real world

**Question:** A species of extremely rare, deep water fish has an extremely long lifespan and rarely have children. If there are a total 821 of this type of fish and their growth rate is 2% each month, how many will there be in half of a year? What will be the population be in 10 years and in 100 years ?

**Answer**

**Step 1:**  $Population = Initial\ population \times (1 + growth\ percentage)^{time\ period\ in\ months}$

Therefore, in this case:

$Population = 821(1,02)^n$ , where  $n =$  number of months

**Step 2 : In half a year = 6 months**

$Population = 821(1,02)^6 = 924$

**Step 3 : In 10 years = 120 months**

$Population = 821(1,02)^{120} = 8\,838$

**Step 4 : in 100 years = 1 200 months**

$Population = 821(1,02)^{1\,200} = 1,716 \times 10^{13}$

Note this answer is also given in scientific notation as it is a very big number.

## 17.4 End of chapter Exercises

1. Simplify as far as possible:

A  $8^{-\frac{2}{3}}$

B  $\sqrt{16} + 8^{-\frac{2}{3}}$

2. Simplify:

(a)  $(x^3)^{\frac{4}{3}}$

(b)  $(s^2)^{\frac{1}{2}}$

(c)  $(m^5)^{\frac{5}{3}}$

(d)  $(-m^2)^{\frac{4}{3}}$

(e)  $-(m^2)^{\frac{4}{3}}$

(f)  $(3y^{\frac{4}{3}})^4$

3. Simplify as much as you can:

$$\frac{3a^{-2}b^{15}c^{-5}}{(a^{-4}b^3c)^{-\frac{5}{2}}}$$

4. Simplify as much as you can:

$$(9a^6b^4)^{\frac{1}{2}}$$

5. Simplify as much as you can:

$$\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16}$$

6. Simplify:

$$x^3\sqrt{x}$$

7. Simplify:

$$\sqrt[3]{x^4b^5}$$

8. Re-write the expression as a power of  $x$ :

$$\frac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{\sqrt[3]{x}}$$



## Appendix A

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